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The Asymptotic Groundstate of SU(3) Matrix Theory

Jens Hoppe and Jan Plefka

Albert-Einstein-Institut
Max-Planck-Institut für Gravitationsphysik
Am Mühlenberg 1, D-14476 Golm, Germany
hoppe,plefka@aei-potsdam.mpg.de

Abstract

The asymptotic form of a $SU(3)$ matrix theory groundstate is found by showing that a recent ansatz for a supersymmetric wavefunction is non-trivial (i.e. non-zero).

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Maximally supersymmetric $SU(N)$ gauge quantum mechanics [1] in $d=9$ has in recent years received much attention due to its close relation[1] to the eleven-dimensional supermembrane [3] in the $N \to \infty$ limit, its description of the dynamics of N D0 branes in superstring theory [4], as well as the M theory proposal of [5]. In these physical interpretations the existence of a unique normalizable zero-energy groundstate [6] is an important consistency requirement. An explicit construction of the vacuum state, though highly desirable, appears to be quite difficult. Another approach is to study the behavior of the wavefunction far out at infinity where the degrees of freedom in the Cartan-subalgebra become free and the remaining degrees of freedom form the zero energy vacuum state of supersymmetric harmonic oscillators [7, 8]. The full asymptotic groundstate was constructed for the $SU(2)$ model in [8], here we consider the $SU(3)$ case. Assuming that the Cartan-subalgebra degrees of freedom are asymptotically governed by a set of free effective supercharges Q_α a proposal was recently made [9] as to which of the harmonic wavefunctions constructed in [10] is annihilated by the Q_α. In this letter we prove the non-triviality of this ansatz.

The asymptotic supersymmetry charge for the $d=9$ SU(3) model reads

$$Q_\alpha = -i\Gamma^a_{\alpha\beta}\left(\theta^1_\beta \frac{\partial}{\partial x^a_1} + \theta^2_\beta \frac{\partial}{\partial x^a_2}\right) \tag{1}$$

where x^a_1, x^a_2 ($a = 1, \ldots, 9$) are the bosonic and $\theta^1_\alpha, \theta^2_\alpha$ ($\alpha = 1, \ldots, 16$) are the fermionic degrees of freedom of the Cartan sector; we work with a real, symmetric representation of the Dirac matrices and our charge conjugation matrix equals unity. It is advantageous to go to the complex variables

$$\begin{aligned} \lambda &= \tfrac{1}{\sqrt{2}}(\theta^1 + i\theta^2) \qquad & z^a &= \tfrac{1}{\sqrt{2}}(x^a_1 + ix^a_2) \\ \lambda^\dagger &= \tfrac{1}{\sqrt{2}}(\theta^1 - i\theta^2) \qquad & \bar{z}^a &= \tfrac{1}{\sqrt{2}}(x^a_1 - ix^a_2)\,. \end{aligned} \tag{2}$$

Note that we have now divided the fermions into creation and annihilation operators, obeying the algebra

$$\{\lambda_\alpha, \lambda^\dagger_\beta\} = \delta_{\alpha\beta}\,, \tag{3}$$

and we define the fermionic vacuum $|-\rangle$ by $\lambda_\alpha|-\rangle = 0$. The completely filled state is denoted by $|+\rangle = \frac{1}{16!}\,\epsilon^{\alpha_1\ldots\alpha_{16}}\,\lambda^\dagger_{\alpha_1}\ldots\lambda^\dagger_{\alpha_{16}}|-\rangle$. Clearly $|-\rangle$ and $|+\rangle$ are $SO(9)$ singlets. However, there is a third $SO(9)$ singlet state

$$|\mathbb{1}\rangle = (\lambda^\dagger\Gamma^{ab}\lambda^\dagger)\,(\lambda^\dagger\Gamma^{bc}\lambda^\dagger)\,(\lambda^\dagger\Gamma^{cd}\lambda^\dagger)\,(\lambda^\dagger\Gamma^{da}\lambda^\dagger)\,|-\rangle \tag{4}$$

in the half-filled sector. It can be shown [10] that there are no further $SO(9)$ singlets. A further symmetry group acting on these states is the Weyl group, the discrete asymptotic remnant of the continous $SU(3)$ of the full system. The Weyl group for $SU(3)$ may be generated by two elements P and C [10], which act on the complex fermions λ and $\lambda^\dagger$ as

$$\begin{aligned} P: &\qquad \lambda \to \lambda^\dagger &\qquad \lambda^\dagger &\to \lambda \\ C: &\qquad \lambda \to e^{-\frac{2\pi i}{3}}\,\lambda &\qquad \lambda^\dagger &\to e^{\frac{2\pi i}{3}}\,\lambda^\dagger\,. \end{aligned} \tag{5}$$

[1] based on [2]

As P interchanges $|+\rangle$ and $|-\rangle$, leaves the eight fermion sector invariant and the three $SO(9)$ singlets are known to form one two dimensional irreducible representation under the Weyl group and one singlet [9] the state $|\mathbb{1}\rangle$ has to be Weyl invariant. This is consistent with C transforming $|\pm\rangle$ into $\exp(\mp\frac{2\pi i}{3})\,|\pm\rangle$. So $|\mathbb{1}\rangle$ is the unique $SO(9)$ *and* Weyl invariant state.

In the complex variables the supersymmetry charge (1) reads $Q_\alpha = -i(\partial\!\!\!/\lambda)_\alpha - i(\bar{\partial}\!\!\!/\lambda^\dagger)_\alpha$, where $\partial_a = d/d\,z^a$ and $\bar{\partial}_a = d/d\,\bar{z}^a$. We seek for an asymptotic groundstate $|\Psi\rangle$ obeying $Q_\alpha|\Psi\rangle = 0$. Note that while Q_α squares to $-\partial\cdot\bar{\partial}$, the condition $\partial\cdot\bar{\partial}\,|\Psi\rangle = 0$ does not imply $Q_\alpha\,|\Psi\rangle = 0$ due to the purely asymptotic considerations, i.e. $|\Psi\rangle$ not being square integrable caused by its singularity at the origin.

Consider now the ansatz for $|\Psi\rangle$

$$|\Psi\rangle = \epsilon^{\alpha_1\ldots\alpha_{16}}\,Q_{\alpha_1}\ldots Q_{\alpha_{16}}\,\frac{1}{(z\cdot\bar{z})^8}\,|\mathbb{1}\rangle\,. \tag{6}$$

$|\Psi\rangle$ is obviously annihilated by Q_α, as Q_α squares to the Laplacian $\partial\cdot\bar{\partial}$ which in turn annihilates the harmonic function $(z\cdot\bar{z})^{-8}$. Note that $|\Psi\rangle$ is $SO(9)\times$Weyl invariant by construction. What remains to be shown, however, is that $|\Psi\rangle$ is *non-vanishing*.

For this we consider the matrix element

$$\begin{aligned}\langle -|\,\Psi\rangle \;&=\; \epsilon^{\alpha_1\ldots\alpha_{16}}\,\langle -|\,[(\partial\!\!\!/\lambda)_{\alpha_1} + (\bar{\partial}\!\!\!/\lambda^\dagger)_{\alpha_1}]\ldots[(\partial\!\!\!/\lambda)_{\alpha_{16}} + (\bar{\partial}\!\!\!/\lambda^\dagger)_{\alpha_{16}}]\\ &\qquad (\lambda^\dagger\Gamma^{ab}\lambda^\dagger)\,(\lambda^\dagger\Gamma^{bc}\lambda^\dagger)\,(\lambda^\dagger\Gamma^{cd}\lambda^\dagger)\,(\lambda^\dagger\Gamma^{da}\lambda^\dagger)\,|-\rangle\,\frac{1}{(z\cdot\bar{z})^8}\,,\end{aligned} \tag{7}$$

which we now need to normal order by making use of the anticommutator relation

$$\{(\partial\!\!\!/\lambda)_\alpha, (\bar{\partial}\!\!\!/\lambda^\dagger)_\beta\} = \delta_{\alpha\beta}\,\partial\cdot\bar{\partial} + \Gamma^{ab}_{\alpha\beta}\,\partial_a\,\bar{\partial}_b\,. \tag{8}$$

From the 2^{16} terms generated from expanding out the brackets in the first line of (7) only those containing 4 $(\bar{\partial}\!\!\!/\lambda^\dagger)$ and 12 $(\partial\!\!\!/\lambda)$ survive. Normal ordering of these terms then yields

$$\begin{aligned}\langle -|\,\Psi\rangle \;&\sim\; \epsilon^{\alpha_1\ldots\alpha_{16}}\,\Gamma^{a_1a_2}_{\alpha_1\alpha_2}\,\bar{\partial}_{a_1}\partial_{a_2}\,\Gamma^{a_3a_4}_{\alpha_3\alpha_4}\,\bar{\partial}_{a_3}\partial_{a_4}\,\Gamma^{a_5a_6}_{\alpha_5\alpha_6}\,\bar{\partial}_{a_5}\partial_{a_6}\,\Gamma^{a_7a_8}_{\alpha_7\alpha_8}\,\bar{\partial}_{a_7}\partial_{a_8}\\ &\qquad \langle -|\,(\partial\!\!\!/\lambda)_{\alpha_9}\ldots(\partial\!\!\!/\lambda)_{\alpha_{16}}\,(\lambda^\dagger\Gamma^{ab}\lambda^\dagger)\,(\lambda^\dagger\Gamma^{bc}\lambda^\dagger)\,(\lambda^\dagger\Gamma^{cd}\lambda^\dagger)\,(\lambda^\dagger\Gamma^{da}\lambda^\dagger)\,|-\rangle\,\frac{1}{(z\cdot\bar{z})^8}\,,\end{aligned}$$

and the final contractions then result in

$$\begin{aligned}\langle -|\,\Psi\rangle \;&\sim\; \epsilon^{\alpha_1\ldots\alpha_{16}}\,\Gamma^{a_1a_2}_{\alpha_1\alpha_2}\,\bar{\partial}_{a_1}\partial_{a_2}\,\Gamma^{a_3a_4}_{\alpha_3\alpha_4}\,\bar{\partial}_{a_3}\partial_{a_4}\,\Gamma^{a_5a_6}_{\alpha_5\alpha_6}\,\bar{\partial}_{a_5}\partial_{a_6}\,\Gamma^{a_7a_8}_{\alpha_7\alpha_8}\,\bar{\partial}_{a_7}\partial_{a_8}\\ &\qquad (\partial\!\!\!/\Gamma^{ab}\partial\!\!\!/)_{\alpha_9\alpha_{10}}\,(\partial\!\!\!/\Gamma^{bc}\partial\!\!\!/)_{\alpha_{11}\alpha_{12}}\,(\partial\!\!\!/\Gamma^{cd}\partial\!\!\!/)_{\alpha_{13}\alpha_{14}}\,(\partial\!\!\!/\Gamma^{da}\partial\!\!\!/)_{\alpha_{15}\alpha_{16}}\,\frac{1}{(z\cdot\bar{z})^8}\,,\end{aligned} \tag{9}$$

where the precise (non-zero) combinatorial coefficient in this relation is not of interest, as we only need to show the non-vanishing of $\langle -|\,\Psi\rangle$. In order to proceed we note that

$$(\partial\!\!\!/\Gamma^{ab}\partial\!\!\!/)_{[\alpha\beta]} = \Gamma^{ab}_{\alpha\beta}\,\partial\cdot\partial + 4\,\partial^{[a}\,\Gamma^{b]c}_{\alpha\beta}\,\partial_c\,. \tag{10}$$

Hence (9) may be reduced to a differential operator in ∂_a and $\bar{\partial}_a$ of degree 16 acting on $(z \cdot \bar{z})^{-8}$ provided we know the precise form of the 16 index tensor $t_{(16)}^{a_1 \dots a_{16}}$

$$t_{(16)}^{a_1 \dots a_{16}} = \epsilon^{\alpha_1 \dots \alpha_{16}} \, \Gamma^{a_1 a_2}_{\alpha_1 \alpha_2} \, \Gamma^{a_3 a_4}_{\alpha_3 \alpha_4} \dots \Gamma^{a_{15} a_{16}}_{\alpha_{15} \alpha_{16}} \, . \tag{11}$$

Clearly $t_{(16)}^{a_1 \dots a_{16}}$ must be expressable in form of a large string of space-indexed δ-functions, the $\epsilon^{a_1 \dots a_9}$ symbol cannot appear. Interestingly enough this tensor also appears in the leading one-loop quantum correction to the M-theory effective action contracted with four Riemann tensors [11]. Its precise form can be computed [12] and is most conveniently written down in a form contracted with an antisymmetric auxiliary tensor X^{ab}

$$\begin{aligned} t_{(16)}^{a_1 \dots a_{16}} \, X^{a_1 a_2} \dots X^{a_{15} a_{16}} \;\; = \;\; & 105 \cdot 2^{19} \Big[-5 \, (\mathrm{tr} X^2)^4 + 384 \, \mathrm{tr} X^8 - 256 \, \mathrm{tr} X^2 \, \mathrm{tr} X^6 \\ & \qquad + 72 \, (\mathrm{tr} X^2)^2 \, \mathrm{tr} X^4 - 48 \, (\mathrm{tr} X^4)^2 \Big] \end{aligned} \tag{12}$$

where the product of X is to be understood in the matrix sense.

The knowledge of $t_{(16)}^{a_1 \dots a_{16}}$ now enables us to finally evaluate (9) using (10), which is still rather involved and most effectively done with the help of the computer algebra system FORM [13]. Our final result reads

$$\langle - | \, \Psi \rangle \sim (\partial^2)^4 \left[\, (\partial \cdot \bar{\partial})^2 - \partial^2 \, \bar{\partial}^2 \, \right]^2 \frac{1}{(z \cdot \bar{z})^8} = (\partial^2)^6 \, (\bar{\partial}^2)^2 \, \frac{1}{(z \cdot \bar{z})^8} \, , \tag{13}$$

which is *non-vanishing* and completes the proof of the non-triviality of (6).

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